Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 17

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Textbook Sections: 11.2

Today's Topic: Introducing Series

Last week, we studied sequences of numbers: $a_1, a_2, a_3, \dots$

This week, we ask what happens if we add all the terms of a sequence together: $a_1 + a_2 + a_3 + \cdots$

For example: What happens if we try to add all the reciprocal powers of 2 together?

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Main Def: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The "sum" of all the terms a_n is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$ (or $\sum a_n$ for short)

and is called an **infinite series**.

Immediately, we should ask if this **makes sense**. For example, what should the values of the following infinite series be?

1. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$
2. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$
3. $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots$

These examples should suggest that determining the sum of an infinite series is a subtle task, requiring caution!

Key Idea: The way we make sense of infinite series is a lot like how we define improper integrals:

1) First consider the **partial sums**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_N = \sum_{n=1}^{N} a_n$$
$$\vdots$$

for every $N \geq 1$, S_N is simply a finite sum of real numbers!

2) Now consider the **sequence of partial sums**

$$\{S_n\}_{n=1}^{\infty}$$

As n gets larger and larger, S_n gets "closer" to capturing the behavior of the full sum.

3) We say $\sum_{n=1}^{\infty} a_n$ **converges** to a value S iff the sequence $\{S_n\}_{n=1}^{\infty}$ of **partial sums converges** to S:

If $\{S_n\}_{n=1}^{\infty}$ does not converge to a finite limit, then we say that $\sum_{n=1}^{\infty} a_n$ is **divergent**.

A brief aside: Why do we care about series?

- Series of **real numbers** as we have just described show up in many places: eg in number theory / cryptography, in finance (in certain price models), and in statistics / probability (in models especially).
- As we will see in due course, series of **functions** are extremely important. eg) **Fourier series** underlie much of audio and image processing, inverse problems, etc. Soon, we'll study a related concept: **Taylor series**.

An extremely important guiding example

Recall our first example $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Notice: a_{n+1} is always $\frac{1}{2}$ of a_n, for all $n \geq 1$. In general, a series where each term is a fixed fraction of the previous term is called a **geometric series**:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \qquad (a \neq 0).$$

In the above example, $a = \frac{1}{2}$ and $r = \frac{1}{2}$. We call $r \in \mathbb{R}$ the **common ratio**.

Let's carefully analyze this series to understand when it converges.

① **$r = 1$:**

$$\sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} a\,1^{n-1} = a + a + a + \cdots$$

Here, $S_N = \sum_{n=1}^{N} a = N \cdot a$. Since $a \neq 0$, $S_N \to \pm\infty$ as $N \to \infty$, so $\sum ar^n$ diverges.

② **$r \neq 1$:**

Here's a sneaky trick: Expand and cancel terms in the following expression:

$$\begin{aligned}(1-r)S_N = S_N - r \cdot S_N &= \sum_{n=1}^{N} ar^{n-1} - r\sum_{n=1}^{N} ar^{n-1} \\ &= (a + ar + \cdots + ar^{N-1}) - r(a + ar + \cdots + ar^{N-1}) \\ &= a + ar + \cdots + ar^{N-1} - (ar + ar^2 + \cdots + ar^N) \\ &= a - ar^N = a(1 - r^N)\end{aligned}$$

and thus

$$S_N = \frac{a(1-r^N)}{1-r}.$$

So, convergence of $\sum_{n=1}^{\infty} ar^{n-1}$ is contingent on the convergence of $\{S_N\}_{N=1}^{\infty}$ with $S_N = \frac{(1-r^N)}{1-r}$.

The convergence of this sequence of partial sums thus hinges on the behavior of r^N as $N \to \infty$.

Recall/Exercise: The sequence $\{r^n\}_{n=1}^{\infty}$ converges iff $-1 < r \leq 1$, and moreover $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } r \in (-1,1). \\ 1 & \text{if } r = 1 \end{cases}$

In this case we have $r \neq 1$, so we know that $\sum_{n=0}^{\infty} ar^n$ converges iff $r \in (-1,1)$. When this is the case:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \frac{a}{1-r}.$$

Examples: ① $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, $a = \frac{1}{2}$, $r = \frac{1}{2}$. $\sum \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$.

② $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} 3 \cdot \frac{1}{3^{n-1}} = \frac{3}{1-1/3} = \frac{9}{2}$

③ $-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} (-1)\left(-\frac{1}{2}\right)^{n-1} = \frac{-1}{1+\frac{1}{2}} = -\frac{2}{3}$

Preview for Next time:

Before class on wednesday, try computing the partial sums for

① $\sum_{n=1}^{\infty} \frac{1}{n}$

② $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$